

CISM governing equations and numerical solution

From Interactive System for Ice sheet Simulation

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Links

Topics mentioned below that have additional detailed discussion at their own pages include:

1. The primary higher-order dynamics scheme used in CISM
2. Boundary conditions for the higher-order dynamics scheme in CISM
3. Solution of the higher-order dynamics scheme in CISM
4. Solving for thickness evolution with a higher-order dynamics scheme

Basics

The main distinction between so-called "higher-order" models and "0-order" (or "shallow ice") models is that higher-order models attempt a closer approximation to solving the non-linear Stokes equations. In general, this usually means incorporating some approximation of horizontal-stress gradients - along-flow stretching or compression and across-flow shearing - in addition to the vertical stress gradients that are accounted for in shallow ice models. This is important for several reasons:

- For parts of the ice sheet that we are the most interested in - e.g. ice streams, ice shelves, and other regions of fast flow - horizontal-stress gradients are as or more important than vertical stress gradients. To model the flow in these regions accurately, higher-order models are required.
- Shallow ice, applied to situations in which there is basal sliding, gives rise to a singularity in the vertical velocity. Models compute the vertical velocity by integrating incompressibility

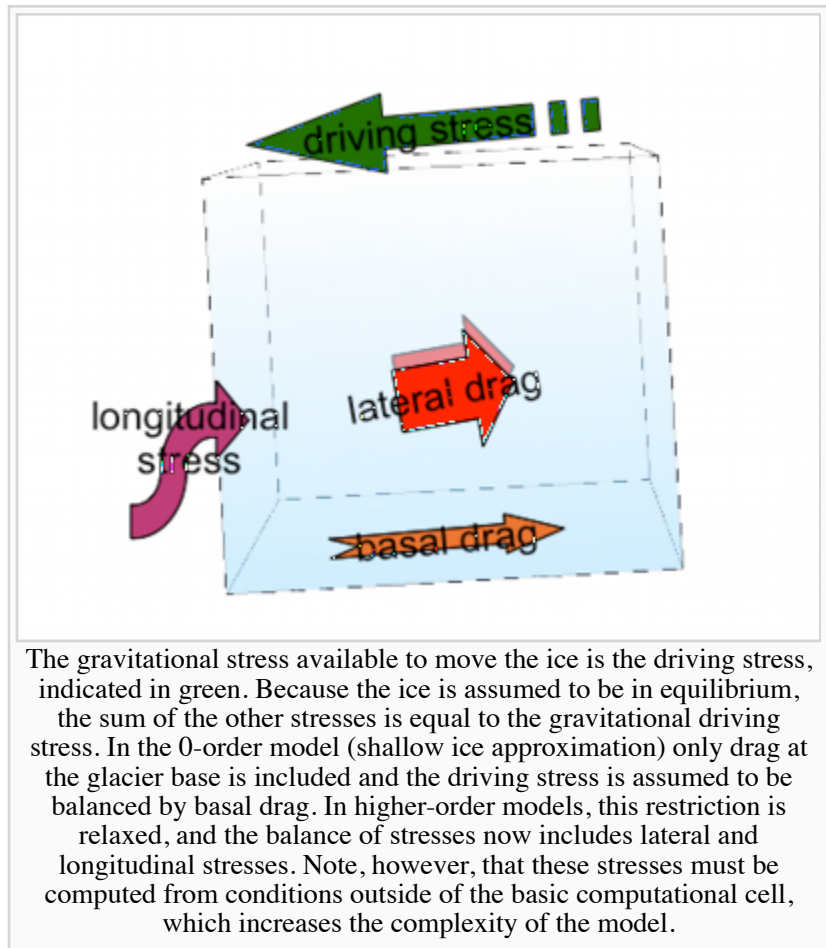
$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

If there is a "jump" from no-sliding in one grid cell to sliding in the next, the horizontal velocity gradients at the bed will be entirely

dependent on the grid spacing; the horizontal gradients (and through incompressibility, the vertical velocity gradient and thus the vertical velocity) will become increasingly large as the grid spacing decreases. Obviously, this is nonsensical and to be avoided.

- Incomplete knowledge of the stresses near the grounding line makes it unlikely that shallow ice models will ever be able to accurately simulate grounding line advance and retreat.
- In some regions of very slow flow, horizontal-stress gradients are important or dominant. For example, ice cores are often recovered at ice divides. Flow modeling is important for interpreting ice core records and using information (such as layer thickness) to infer something about the past flow history in the region. In order to model that flow correctly, one must include horizontal stresses (At an ice divide the surface slope is ~ 0 , in which case vertical stress gradients that drive deformation in 0-order models are also ~ 0 . In reality, deformation is not 0 at ice divides it is simply controlled by horizontal stretching).

The term "higher-order" comes from scaling analyses of the Stokes equations for which a scaling parameter $\lambda = H/L$ - the ratio of the thickness to the horizontal length scale of interest - is used to assign importance to the various terms (see the references at the bottom of **this** page for a more detailed discussion). Shallow ice models retain only terms of order 0 while "higher-order" models also retain terms of order 1 (and possibly greater).



Full constitutive relation

$$\mathbb{T} - p\mathbb{I} = 2\mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y} & \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ \frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) & \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) & \frac{\partial w}{\partial z} \end{pmatrix}$$

Reduced constitutive relation

$$\mathbb{T} - p\mathbb{I} = 2\mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{1}{2}\frac{\partial u}{\partial z} \\ \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y} & \frac{1}{2}\frac{\partial v}{\partial z} \\ \frac{1}{2}\frac{\partial u}{\partial z} & \frac{1}{2}\frac{\partial v}{\partial z} & -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \end{pmatrix}$$

The constitutive equations before and after the terms of order higher than one are dropped. Modified terms are indicated in blue. Note that all derivatives of w are gone, eliminating a degree of freedom from the equations. Another degree of freedom can be eliminated by integrating the simplified vertical stress balance in the vertical and taking account of incompressibility, as shown **here**.

Available schemes

The most basic and fundamental higher-order scheme is a solution to the full non-linear Stokes equations. Because of the computational burden, many 3d, large-scale models solve lower-order approximations to the Stokes equations. However, a number of groups are making significant advances in this area and it seems likely that 3d, nonlinear-Stokes ice sheet models being used in climate-model applications is not far away (e.g. see the ELMER Ice (http://www.csc.fi/english/csc/news/news/elmer_eng2008-1-14) effort). An upcoming DOE effort will focus on implementing a nonlinear Stokes model on unstructured grids in CISM.

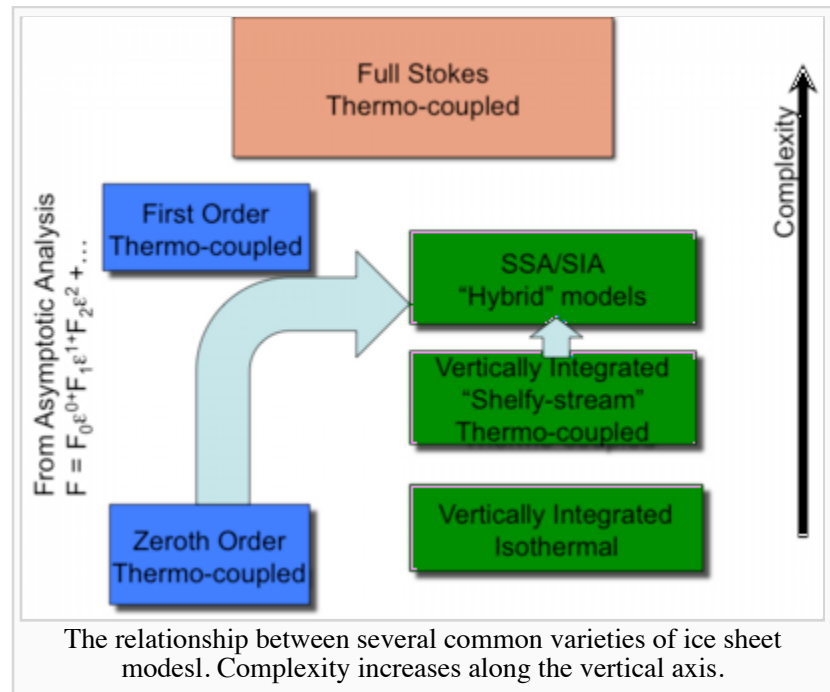
- Probably the most long-lived higher-order approximation in glaciology is the "shallow-shelf approximation" (or SSA) describing flow within an ice shelf. It was made popular by Doug MacAyeal in the 80's and 90's. Its main disadvantage is that it is not fully 3d, as it assumes uniform velocity throughout the ice thickness driven only by horizontal stress gradients. It is, however, adequate for describing fast flow in many parts of the ice sheets, such as on ice shelves and along some ice streams. In this case, not resolving vertical gradients is a computational advantage.
- The SSA equations are actually a depth-averaged form of a more general higher-order model, which is commonly referred to as the Blatter-Pattyn model. This model has been around since the mid 90's and has become increasingly popular ever since. Blatter-Pattyn dynamics are currently implemented for the higher-order dynamics in the publicly available version of CISM. For this coarse, we will do several sets of experiments using this higher-order scheme.
- Several "hybrid" schemes exist that are computationally cheaper than the Blatter-Pattyn model.

These combine solutions to the shallow ice approximation (for resolving vertical gradients) and the SSA approximation (for resolving horizontal gradients) in some clever way so that a fully 3d solution is obtained. It isn't yet known how well these model solutions compare to fully 3d models, or if one approach (hybrid vs. fully 3d solution) is superior to the other. David Pollard of Penn State and Ed Bueler of Univ. of Alaska Fairbanks currently run large-scale implementations of this type of model (see references below).

Practical differences between shallow ice models and higher-order models

By "practical differences", we mean (1) how do we deal with solving the momentum equations in each case (the dynamics, or stress balance) and (2) how do we use the relevant information we derive from that solution (the kinematics, or velocity fields) to evolve the ice sheet geometry in time? There are large differences in how both of these issues are handled - in shallow ice models versus in higher-order models - for two main reasons:

- The numerical solution of the dynamical equations is fundamentally different in each case. For the shallow ice case, we need only local information (slope and thickness) to solve for the velocity as a function of depth in a single column of ice. We do this pointwise for every location on our model domain (in map view), which is a relatively easy numerical problem; each column of velocities leads to a banded coefficient matrix that is relatively easy to invert (to solve for the velocities). This problem is also what we might call "embarrassingly parallel"; in theory, each column of unknown velocities results in its own tridiagonal matrix, which could be inverted for on its own processor. For higher-order models, however, we cannot do this since the solution at any point also depends on the solution at neighboring points (in map plane). The velocity at any point depends on non-local information, leading to an elliptic system of equations, and every velocity must be solved for simultaneously with every other velocity. The result is a much larger system of equations to solve, which is a more difficult numerical problem to solve on one processor and a much more difficult problem to solve on multiple processors. Because large-scale applications of higher-order models (e.g. whole-ice sheet models and coupling with climate models) will require efficient solution and parallelization techniques, this is a very active area of current research.



- The equations governing dynamics AND evolution in a shallow ice model can be recast together as a single, non-linear, diffusion equation for ice thickness. A single system of equations is solved to calculate the velocity field and evolve the ice sheet geometry. For higher-order models, we must first solve the momentum balance equations to obtain the velocity field. Then, we need to use some other scheme to evolve the ice thickness.

Both of these differences mean that a model based on shallow ice physics may be built in a fundamentally different way than one based on higher-order physics. Most of the development work on CISM during the past few years has had to do with "upgrading" the model so that it can be used effectively and efficiently with higher-order dynamics schemes.

The equations describing a (2d) higher-order flow that is vertically integrated are

$$\begin{aligned}\frac{\partial}{\partial x} \left(2\eta H \left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left(\eta H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) &= \rho_w g H \frac{\partial s}{\partial x} \\ \frac{\partial}{\partial y} \left(2\eta H \left(2\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \right) + \frac{\partial}{\partial x} \left(\eta H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) &= \rho_w g H \frac{\partial s}{\partial y}\end{aligned}$$

The equations describing a (3d) higher-order flow that is NOT vertically integrated are

$$\begin{aligned}\frac{\partial}{\partial x} \left(2\eta \left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left(\eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial z} \left(\eta \frac{\partial u}{\partial z} \right) &= \rho_w g \frac{\partial s}{\partial x} \\ \frac{\partial}{\partial y} \left(2\eta \left(2\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) \right) + \frac{\partial}{\partial x} \left(\eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial z} \left(\eta \frac{\partial v}{\partial z} \right) &= \rho_w g \frac{\partial s}{\partial y}\end{aligned}$$

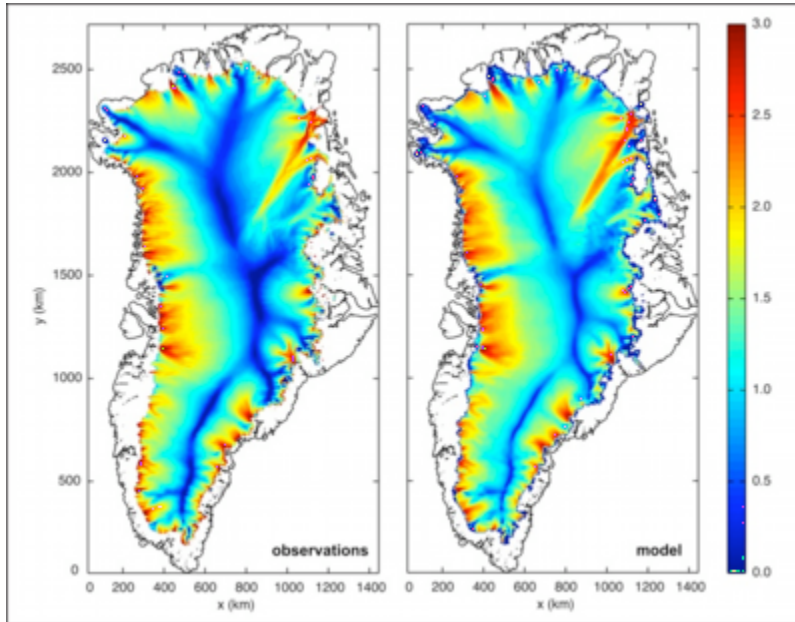
There are three differences that you should note

1. The vertically integrated model includes H , or ice thickness in each term. This is a reflection of the integration and does not appear in the first order equations.
2. Accounting for the thickness not appearing, the only other difference is the presence of a vertical diffusion of horizontal velocities. This is the the third term on the left in the above equations.
3. The first order equations must be solved on each of a set of horizontal "layers". Layers communicate with each other through the diffusion term.

Both sets of equations are non-linear elliptical equations and much of the same "technology" can be used solve them. Additional complications come in when we account for boundary conditions at the lateral margins of the domain and at the upper and lower surfaces of the flow (note that for the vertically

integrated flow, there are no explicit boundary conditions for the upper and lower surfaces; they are accounted for and "incorporated" during the vertical integration).

Higher-order CISM



The higher-order dynamics scheme currently implemented in CISM is discussed in more detail **here**. Boundary conditions are discussed **here** and the numerical solution is discussed **here**.

A very useful higher-order model intercomparison project (**ISMIP-HOM** (<http://homepages.ulb.ac.be/~fpattyn/ismip/>)) was organized by Frank Pattyn from the Université Libre de Bruxelles. That project, which resulted in a set of "benchmark" experiments for higher-order models, is reported on formally in Pattyn et. al (2008) (<http://www.the-cryosphere.net/2/95/2008>

[/tc-2-95-2008.html](#)) .

As a **first exercise**, we will build a recent version of CISM on your local computer. In a **second set of exercises** we will take advantage of a nice feature of CISM and run the higher-order model through an automated test suite that allows us to compare diagnostic solutions from CISM with output from the ISMIP-HOM benchmark study. In the **third set of exercises**, we will add a new thickness-change calculation module to CISM. This will allow us to evolve some of our previous diagnostic solutions forward in time to a steady-state.

References

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